1. The pseudorange equation.

The range is not measured directly. The receiver ‘measures’ the time of reception of a satellite signal using a clock inside the receiver, and determines the time of transmission by a clock inside the satellite. The time difference (transit time) is multiplied by the speed of light (c) to obtain range.

- The satellite clock is not perfectly synchronized to the GPS system time. The satellite clock deviation $C_i$ can be calculated using the clock parameters embedded in the ‘navigation message’, see chapter 2.
- The receiver clock is not synchronized to GPS time. The receiver clock deviation $C_u$ can not be calculated directly, but is calculated as a by-product of the user position calculation.

The atmosphere slows the satellite signal.
- The ionospheric delay $I_i$ is estimated using the navigation message iono parameters (chapter 2).
- The tropospheric delay $T_i$ is estimated using a model (see chapter 2)

Due to the above-mentioned imperfections the range measurement is called ‘Pseudorange’. Before entering the user position calculation, the pseudoranges are first corrected for a, c, and d above:

$$PR_{i,u}^{\text{corrected}} = PR_{i,u}^{\text{measured}} + C_i - I_i - T_i$$

The ‘true’ range relates to the corrected pseudorange as

$$R_{i,u} = PR_{i,u}^{\text{corrected}} + C_u,$$

or

$$R_{i,u} - C_u = PR_{i,u}^{\text{corrected}}$$

2. The carrier range equation

In the following sections $N$ is the number of common satellites tracked by a reference receiver and a user receiver.

The carrier range $CR_{i,u}$ (in meters) is obtained by multiplying the carrier phase measurement $\Phi_{i,u}$ with the carrier wavelength $\lambda_1$ ($\lambda_1$ equals the speed of light $c$ divided by the first carrier frequency, and is about 0.20 m).

Like the pseudorange measurement, the carrier range measurement $CR_{i,u}$ (unit: meters) is not perfect. The carrier range must be corrected for satellite clock offset and atmospheric delay in analogy with [1-2]:

$$CR_{i,u}^{\text{corrected}} = CR_{i,u}^{\text{measured}} + C_i + I_i - T_i, \quad (i = 1, \ldots, N).$$

The corrections are calculated using the user position from e.g. a single difference pseudorange position calculation, and the user receiver measurement time. Note the opposite sign of $I_i$. The carrier phase experiences an advance when travelling through the ionosphere, while the code on the carrier experiences a delay of the same magnitude.

As mentioned in chapter 6, the carrier range measurement is very precise, but lacks information on the initial range, or better the initial number of carrier wavelengths between satellite and receiver. Add the unknown number of wavelength $N_{i,u}$ (converted to m), and the carrier range equation becomes analog to [1-3]:

$$R_{i,u} - C_u = CR_{i,u}^{\text{corrected}} + \lambda_1 * N_{i,u}$$

The term $CR_{i,u}^{\text{corrected}} + \lambda_1 * N_{i,u}$ is equal to $PR_{i,u}$ and in theory $N_{i,u}$ could be calculated. In practice, the error in $PR_{i,u}$ is at the meter level, so the uncertainty in $N_{i,u}$ is many wavelengths and $N_{i,u}$ can not reliably rounded towards its true integer value.
In the following it shall be shown that by forming differences, the unknown number of carrier wavelengths can be solved.

The (unknown) receiver clock offset $C_u$ can be removed by forming (for each receiver) differences between satellites. Appoint one satellite as reference (usually one takes the high elevation satellite) and form the difference:

\[
R_u^i - R_u^1 = CR_u^i (c) - CR_u^1 (c) + \lambda_i \ast (N_u^i - N_u^1), \quad (i = 2, \ldots, N, \text{ref sat} = 1, c = \text{corrected})
\]

If the true ranges $R_u^i$ were known (initiate the receiver at a surveyed location) one could calculate $N_u^i - N_u^1$. However, the remaining errors in the measured carrier ranges (satellite clock model errors, iono delay model error, and tropo delay model error) are still at meter level.

Differencing between receivers removes the correlated errors.

For a reference receiver equation [2-1] becomes:

\[
CR_r^i (c) = CR_r^1 (m) + C_i + I_i - T_i, \quad (i = 1, \ldots, N).
\]

The corrections are calculated using the reference position and the reference receiver measurement time.

[2-3] Becomes:

\[
R_r^i - R_r^1 = CR_r^i (c) - CR_r^1 (c) - \lambda_i \ast (N_r^i - N_r^1),
\]

and differencing between receivers gives:

\[
(R_u^i - R_u^1) - (R_r^i - R_r^1) = (CR_u^i (c) - CR_u^1 (c)) - (CR_r^i (c) - CR_r^1 (c)) - \lambda_i \ast ((N_u^i - N_u^1) - (N_r^i - N_r^1)),
\]

or

\[
[2-4] \quad R_{ur}^{i1} = CR_{ur}^{i1} - \lambda_i \ast N_{ur}^{i1} \quad (i = 2, \ldots, N, \text{ref sat} = 1)
\]

with

\[
R_{ur}^{i1} = (R_u^i - R_u^1) - (R_r^i - R_r^1) \\
CR_{ur}^{i1} = (CR_u^i (c) - CR_u^1 (c)) - (CR_r^i (c) - CR_r^1 (c)) \\
N_{ur}^{i1} = ((N_u^i - N_u^1) - (N_r^i - N_r^1))
\]

In the double difference carrier range $CR_{ur}^{i1}$ the clock offsets, nominal atmospheric delay, and correlated errors are removed. Unmodelled errors such as carrier range multipath and noise are at mm to cm level.

Equation [2-4] can be used in two ways:
1. If the user receiver is initiated at a surveyed location the (combined) ambiguity terms $N_{ur}^{i1}$ can be calculated and rounded towards the nearest integer number.
2. Once the integer ambiguities are known, the user can start moving, and the user position can be calculated with extreme high accuracy (mm to cm level).

Initiation of a user (aircraft) receiver at a surveyed position is often impractical. Also, if a receiver looses temporary lock to satellites (which occurs often in the airborne environment) the integer ambiguity may need to be initialized again. Determination of the integer ambiguities ‘on the fly’ is the challenging task. Position calculation is trivial once the integers are determined.

In the next section a practical implementation of the double difference pseudorange equation is derived. Then an analog form for the double difference carrier range equation is given.

3. The double difference pseudorange equation
Equation [2-4] is valid only if the carrier phases of both receiver are measured at the same moment (to within a few microseconds). Not all receivers measure at the full GPS second, see section 3.1 above. For practical implementation the double difference pseudorange equation is set up in a different form.

Starting with the reference receiver, correct the pseudoranges according [1-2]:

\[ PR_i^r(c) = PR_i^r(m) + C_i - T_i^r, \quad (i = 1, \ldots, N, \text{c = corrected, m = measured}) \]

Next, expand [1-3] with bias term \( B_i^r \), containing the correlated error terms to be removed by double differencing:

\[ R_i^r - C_r = PR_i^r(c) + B_i^r, \quad (i = 2, \ldots, N) \]
\[ R_1^r - C_r = PR_1^r + B_1^r, \quad (1 = \text{reference satellite}) \]

Form differences between satellites:

\[ (R_i^r - R_1^r) = (PR_i^r(c) - PR_1^r(c)) + B_{i1}^r, \quad (i = 2, \ldots, N) \]

In \( B_{i1}^r \), the change in range due to satellite movement, the reference receiver clock offset, the satellite clock offsets and the nominal atmospheric delay terms are removed. Hence \( B_{i1}^r \) can easily be interpolated to the measurement time of the user receiver.

Now, correct the pseudorange measurements of the user receiver:

\[ PR_i^u(c) = PR_i^u(m) + C_i - T_i^u, \quad (i = 1, \ldots, N, \text{c = corrected, m = measured}) \]

And expand for the user receiver [1-3] with a bias term:

\[ (R_i^u - R_1^u) = (PR_i^u(c) - PR_1^u(c)) + B_{i1}^u, \quad (i = 2, \ldots, N) \]

By definition, \( B_{i1}^u = B_{i1}^r \). Now the left-hand side of [3-2] contains the unknown user coordinates, while the right hand side contains known quantities only. Note the difference with [1-3]: the (combined) receiver clock offset has disappeared.

Expand \( R_i^r \) and \( R_i^u \) in Taylor series, the differential equation becomes:

\[ (R_i^r(0) + ((X_i - X_u(0)) / R_i^u(0)) \cdot \Delta X + ((Y_i - Y_u(0)) / R_i^u(0)) \cdot \Delta Y + ((Z_i - Z_u(0)) / R_i^u(0)) \cdot \Delta Z) - (R_1^r(0) + ((X_1 - X_u(0)) / R_1^u(0)) \cdot \Delta X + ((Y_1 - Y_u(0)) / R_1^u(0)) \cdot \Delta Y + ((Z_1 - Z_u(0)) / R_1^u(0)) \cdot \Delta Z) = (PR_i^r(c) - PR_1^r(c)) + B_{i1}^r, \quad (i = 2, \ldots, N) \]

or

\[ A'_j \cdot D_j = L'_j, \quad (i = 2, \ldots, N, j = 1, \ldots, 3) \]

with

\[ A'_1 = ((X_i - X_u(0)) / R_i^u(0) - (X_1 - X_u(0)) / R_1^u(0)) \]
\[ A'_2 = ((Y_i - Y_u(0)) / R_i^u(0) - (Y_1 - Y_u(0)) / R_1^u(0)) \]
\[ A'_3 = ((Z_i - Z_u(0)) / R_i^u(0) - (Z_1 - Z_u(0)) / R_1^u(0)) \]
\[ D_j = (\Delta X, \Delta Y, \Delta Z) \]
\[ L'_j = (PR_i^r(c) - PR_1^r(c)) - (R_i^r(0) - R_1^r(0)) + B_{i1}^r \]
The solution of [3-3] is:

\[ D = (A^*Q^{-1}A)^{-1} * (A^*Q^{-1}) * L \]

with \( Q \) the variance covariance matrix. The double difference pseudorange measurements are correlated, hence the off-diagonal elements of \( Q \) are not zero. For simplicity assume the undifferenced pseudorange variance \( \nu \) to be equal for all range measurements (a reasonable guess). The diagonal elements of \( Q \) have the value \( 4 \nu \) and the off-diagonal elements have the value \( 2 \nu \) (Ref. TBS).

Start the iterative solution with \( X_u(0) = X_r, \ Y_u(0) = Y_r, \ Z_u(0) = Z_r \) and repeat the iterations until \( \Delta X, \Delta Y, \Delta Z \) are close to zero.

Note. \( L_i \) contains all unmodelled errors of the measurements by both receivers of the reference satellite and \( i \). It will thus be larger than the error in the single difference pseudorange equation and the calculated user position will be less accurate. This restricts the practical use of the double difference pseudoranges. The above derivation however serves as an introduction into the derivation of the double difference carrier range equations, and will be used later when estimating the double difference integer ambiguities.

### 4. The double difference carrier range equation

Analog to section 2 and 3 correct the carrier ranges of the reference receiver:

\[ CR_i^r (c) = CR_i^r (m) + C^i + I^r_i - T^r_i \] \( (i = 1, \ldots, N) \)

The carrier range equation is:

\[ R_i^r - C_i = CR_i^r (c) + \lambda_1 * N_i^r + B_i^r \]

Form differences between satellites:

\[ [4-1] \quad (R_i^r - R_1^r) = (CR_i^r (c) - CR_1^r (c)) + \lambda_1 * (N_i^r - N_1^r) + B_i^r_i, \quad (i = 2, \ldots, N) \]

and define \( B_i^r_i \) as:

\[ [4-2] \quad B_i^r_i = (R_i^r - R_1^r) - (CR_i^r (c) - CR_1^r (c)), \quad (i = 2, \ldots, N) \]

In \( B_i^r_i \) the change in range due to satellite movement, the reference receiver clock offset, the satellite clock offsets and the nominal atmospheric delay terms are removed. Hence \( B_i^r_i \) can easily be interpolated to the measurement time of the user receiver (as long as no receiver loss of lock is experienced).

With \[4-1\] and \[4-2\] \( B_i^r_i \) is expressed as:

\[ [4-3] \quad B_i^r_i = \theta_i^c * (N_i^r - N_i^r) \]

Now for the user receiver:

\[ CR_i^u (c) = CR_i^u (m) + C^i + I_u - T_u \] \( (i = 1, \ldots, N) \)

\[ [4-4] \quad (R_i^u - R_1^u) = (CR_i^u (c) - CR_1^u (c)) + \lambda_1 * (N_i^u - N_1^u) + B_i^u_i, \quad (i = 2, \ldots, N) \]

Substitute \( B_i^u_i = B_i^r_i - \lambda_1 * (N_i^r - N_i^r) \) into \[4-4\]:

\[ (R_i^u - R_1^u) = (CR_i^u (c) - CR_1^u (c)) + \lambda_1 * (N_i^u - N_i^u) + \theta_i^c - \lambda_1 * (N_i^r - N_i^r) \]

or
\[ (R_i - R_1) = (CR_i (c) - CR_1 (c)) + \lambda_i N_{iu} + \beta_{ir} \quad (i = 2, \ldots, N) \]

Once the double difference integer ambiguities \( N_{iu} \) are known, the solution of [4-5] is trivial and proceeds as in section 3 above. For simplicity assume the undifferenced carrier range variances equal with value \( \upsilon \).

5. Ambiguity fixing

5.1 Introduction

The two sets of \( N-1 \) equations [3-2] and [4-5] (total \( 2N-2 \) equations)

\[ \begin{align*}
  (R_i - R_1) &= (PR_i (c) - PR_1 (c)) + B_{ir}, \\
  (R_i - R_1) &= (CR_i (c) - CR_1 (c)) + \lambda_i N_{iu} + \beta_{ir} \quad (i = 2, \ldots, N)
\end{align*} \]

contain 3 unknown position coordinates and \( N-1 \) unknown double difference integer ambiguities. When tracking 4 common satellites we have 6 equations and 6 unknowns. The set of equations can be solved, but due to the relatively large errors in the pseudorange measurements the error in the ambiguities will usually prohibit rounding towards the correct integer.

For 5 satellites we have 8 equations and 7 unknowns, for 6 satellites 10 equations with 8 unknowns, etc. The over determined set of equations might make reliable rounding of the ambiguities possible.

Obtaining an over determined set of equations could be achieved in an alternative way: add measurements on another moment in time (epoch). One would think that this would raise the number of unknown position coordinates by a factor of two: the coordinates at the first epoch and the coordinates at the second epoch. The precise phase measurements however allow precise calculation of the user position at the second epoch relative to the first epoch without a priori knowledge of the integer ambiguities. The precise position difference is used to project back in time the second measurement to the first measurement, keeping the number of unknown position coordinates at 3.

Thus for 4 satellites we obtain 12 equations with 6 unknowns, for 5 satellites 16 equations with 7 unknowns, etc. And measurements at more epochs can be added. In general with measurements at \( M \) epochs to \( N \) satellites the \( M*(2N-2) \) equations contain \( N+2 \) unknowns.

A third way to raise the over determination is by making range measurements on the second GPS frequency. Advanced (multi 10 k$) receivers measure also pseudoranges and carrier phases on the second frequency. Using these measurements as described in chapter 8 (and change \( \lambda_1 \) into \( \lambda_2 \)) doubles the number of equations, doubles the number of unknown integer ambiguities, but keeps the number of unknown position coordinates on 3.

The required number of epochs to find the correct the integer ambiguities depends on many factors, ao.:

- the number of satellites per epoch,
- the quality of the pseudorange and carrier range measurements (receiver noise and multipath),
- the satellite geometry,
- the change in satellite geometry between epochs,
- the availability of range measurements on the second frequency,
- the efficiency of the method to determine ambiguities.

Solving [5-1] is a basic (least squares) method to determine ambiguities (and precise position coordinates) on the fly. A number of more efficient methods exist, see Refs TBS.
In any method two steps are present: calculate the ambiguities, and validate the ambiguities, although 1 and 2 may be solved together. If the validation fails, more measurements have to be added until validation succeeds.